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Two coupled matrices: eigenvalue correlations and spacing functions

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Abstract. For two $n \times n$ Hermitian matrices A and B, with joint probability density proportional to $\exp[-\operatorname{tr}\{U(A) + V(B) + 2cAB\}]$, where U and V are polynomials, a method is given to calculate all correlation, cluster and spacing functions of the eigenvalues of either one or both matrices. The method relies on the introduction of two sets of bi-orthogonal polynomials with non-local weights. In a linear chain of coupled matrices, if one looks for the statistical properties of the eigenvalues of only one matrix (two matrices), situated anywhere in the chain, then we can proceed as a one-matrix (two-matrices) problem.

1. Introduction

It is known that the joint probability density $\exp(-\operatorname{tr} U(A))$ for the elements of a (real symmetric, Hermitian or quaternion real self-dual) matrix A gives rise to the joint probability density [1]

$$F(x) = f_1 \exp\left\{-\sum_{j=1}^{n} U(x_j)\right\}|\Delta(x)|^\beta \tag{1.1}$$

for its real eigenvalues $x_1, \ldots, x_n$. In equation (1.1), f_1 is a normalization constant

$$\Delta(x) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j) \tag{1.2}$$

and $\beta = 1, 2$ or 4 depending on whether A is real symmetric, Hermitian or quaternion real self-dual. From equation (1.1), one can compute the m-point correlation function

$$R_m(x_1, \ldots, x_m) = \frac{n!}{(n-m)!} \int F(x)\, dx_{m+1} \ldots dx_n \tag{1.3}$$

by introducing appropriate orthogonal or skew-orthogonal polynomials [2]. Here, and in what follows, all the integrals will be taken from $-\infty$ to ∞ unless explicitly indicated otherwise. For two coupled matrices A and B with joint probability density

$$\exp[-\operatorname{tr}\{U(A) + V(B) + 2cAB\}] \tag{1.4}$$

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for their elements, the joint probability density for their eigenvalues is known only in the case when A and B are Hermitian matrices [3–5]. It is

$$F(x; y) = f_2 \exp\left\{-\sum_{j=1}^{n}\{U(x_j) + V(y_j)\}\right\} \det\left[e^{-2cx_i y_j}\right] \Delta(x)\Delta(y) \tag{1.5}$$

where $x_1, \ldots, x_n$ are the eigenvalues of A and $y_1, \ldots, y_n$ are those of B. The factor f_2 is again a normalization constant.

For some applications, it may be of interest to calculate the correlation functions [6]

$$R_{p,q}(x_1, \ldots, x_p; y_1, \ldots, y_q) = \frac{n!}{(n-p)!}\frac{n!}{(n-q)!}\int F(x; y)\,dx_{p+1}\ldots dx_n\,dy_{q+1}\ldots dy_n \tag{1.6}$$

where $0 \leqslant p, q \leqslant n$. The case $p = q = 0$, giving constant normalization, can be disposed of by introducing orthogonal polynomials with non-local weight [4]. The same method works when p or $q = 0$. The correlation function $R_{p,0}(x_1, \ldots, x_p)$ is a $p \times p$ determinant from which the cluster function immediately follows. The probability $E(r, I)$ that an interval I contains exactly r eigenvalues of matrix A (the eigenvalues of B being unobserved) is given as the rth partial derivative of an $n \times n$ determinant. Taking the limit $n \to \infty$ is straightforward provided one knows the corresponding limit for the orthogonal polynomials.

We will consider here the more difficult case $p > 0$, $q > 0$. Our method works for general p and q, but we do not have a compact manageable form. The method gets progressively more tedious as p and q increase. We will explicitly give the functions $R_{p,q}$ for $p, q \leqslant 2$. The situation is much simpler for the probability $E(r, I_1; s, I_2)$ that an interval I_1 contains exactly r eigenvalues of matrix A and an interval I_2 contains exactly s eigenvalues of matrix B. It is given as a certain partial derivative of an $n \times n$ determinant. Taking the limit $n \to \infty$ is again straightforward.

In a linear chain of several coupled matrices, if one considers the correlations or spacing functions of the eigenvalues of one particular matrix, disregarding the eigenvalues of all the other matrices, then their formal structure is the same as those in the one-matrix case. If one considers mixed correlations for the eigenvalues of any two matrices in the chain, disregarding the eigenvalues of all the others matrices, then their formal structure is the same as those for eigenvalues of two coupled matrices.

2. Orthogonal polynomials with non-local weight

Let us indicate briefly how one can evaluate integral (1.6) when either p or $q = 0$. Let $q = 0$. As the integration is performed over all variables $y_1, \ldots, y_n$ and as $\det[e^{-2cx_i y_j}]$ and $\Delta(y)$ are both antisymmetric under the exchange of any two of the variables y_j, we can replace $\det[e^{-2cx_i y_j}]$ by its diagonal term. Thus

$$R_{p,0}(x_1, \ldots, x_p) \propto \int \exp\left\{-\sum_{j=1}^{n}(U(x_j) + V(y_j))\right\} \det\left[e^{-2cx_i y_j}\right] \times \Delta(x)\Delta(y)\,dx_{p+1}\ldots dx_n\,dy_1\ldots dy_n \tag{2.1}$$

$$= n!\int \prod_{j=1}^{n} w(x_j, y_j)\Delta(x)\Delta(y)\,dx_{p+1}\ldots dx_n\,dy_1\ldots dy_n \tag{2.2}$$

where

$$w(x, y) = \exp(-U(x) - V(y) - 2cxy). \tag{2.3}$$

Now, as usual,

$$\Delta(x) = \det[x_i^{j-1}] \propto \det[P_{j-1}(x_i)] \tag{2.4}$$

where $P_j(x) = k_j x^j + \cdots$, $k_j \neq 0$ is any polynomial of degree j. Similarly

$$\Delta(y) = \det[y_i^{j-1}] \propto \det[Q_{j-1}(y_i)] \tag{2.5}$$

where $Q_j(x) = k'_j x^j + \cdots$, $k'_j \neq 0$ is any polynomial of degree j. Choose polynomials $P_j(x)$ and $Q_j(x)$ such that

$$\int w(x, y) P_j(x) Q_k(y)\, \mathrm{d}x\, \mathrm{d}y = \delta_{j,k}. \tag{2.6}$$

This is possible provided that the matrix of moments $[\int w(x, y) x^j y^k\, \mathrm{d}x\, \mathrm{d}y]_{j,k=0,\ldots,m}$ is non-singular for all $m \geqslant 0$.

Let us define

$$\bar{Q}_k(x) = \int w(x, y) Q_k(y)\, \mathrm{d}y. \tag{2.7}$$

Then

$$\begin{aligned}
&\int \prod_{j=1}^{n} w(x_j, y_j) \Delta(x) \Delta(y)\, \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&\qquad \propto \det[P_{j-1}(x_i)] \int \prod_{j=1}^{n} w(x_j, y_j) \det[Q_{j-1}(y_i)]\, \mathrm{d}y_1 \ldots \mathrm{d}y_n \\
&\qquad = \det[P_{j-1}(x_i)] \det[\bar{Q}_{j-1}(x_i)].
\end{aligned} \tag{2.8}$$

Let

$$\sum_{j=0}^{n-1} P_j(x_i) \bar{Q}_j(x_k) = K_n(x_i, x_k). \tag{2.9}$$

Then, in view of equations (2.6) and (2.7), we have

$$\int K_n(x, z) K_n(z, \xi)\, \mathrm{d}z = K_n(x, \xi) \tag{2.10}$$

$$\int K_n(x, x)\, \mathrm{d}x = n \tag{2.11}$$

and therefore, according to a theorem by Dyson (see, for example, [5], ch 8, section 8.10)

$$\int \det[K_n(x_i, x_j)]_{i,j=1,\ldots,m}\, \mathrm{d}x_m = (n - m + 1) \det[K_n(x_i, x_j)]_{i,j=1,\ldots,m-1}. \tag{2.12}$$

Hence, from equation (2.8)

$$\int \mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_{p+1}\int \mathrm{d}y_1\ldots\mathrm{d}y_n\prod_{j=1}^{n} w(x_j,y_j)\Delta(x)\Delta(y)$$

$$\propto \int \mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_{p+1}\ \det[P_{j-1}(x_i)]\det[\bar{Q}_{j-1}(x_i)]$$

$$= \int \mathrm{d}x_n\ldots\mathrm{d}x_{p+1}\det[K_n(x_i,x_j)]_{i,j=1,\ldots,n}$$

$$= (n-p)!\det[K_n(x_i,x_j)]_{i,j=1,\ldots,p}. \tag{2.13}$$

Taking care of all the constants properly, one has

$$R_{p,0}(x_1,\ldots,x_p) = \det[K_n(x_i,x_j)]_{i,j=1,\ldots,p}. \tag{2.14}$$

The cluster function (or the cumulant or connected correlator!) $T_{p,0}(x_1,\ldots,x_p)$ (for its definition, see, for example, [1], ch 5) can be written as

$$T_{p,0}(x_1,\ldots,x_p) = \sum K_n(x_1,x_2)K_n(x_2,x_3)\ldots K_n(x_p,x_1) \tag{2.15}$$

where the sum is taken over all distinct $(p-1)!$ cyclic permutations of the indices $(1,2,\ldots,p)$.

Similar considerations will give the correlation function $R_{0,q}(y_1,\ldots,y_q)$ and the cluster function $T_{0,q}(y_1,\ldots,y_q)$.

If we disregard the eigenvalues of B, $y_1,\ldots,y_n$, then one can also compute the level spacing functions of $x_1\ldots,x_n$. To obtain the probability $E(0,I)$ that an interval I does not contain any of the x_j, one integrates over all x_j outside the interval I; which, according to the Binet–Cauchy formula (see, for example, [5], ch 3), can be written as

$$\frac{1}{n!}\left(\int - \int_I\right)\det[K_n(x_i,x_j)]\,\mathrm{d}x_1\ldots\mathrm{d}x_n = \det[G_{i,j}]_{i,j=0,\ldots,n-1} \tag{2.16}$$

where

$$G_{i,j} = \left(\int - \int_I\right)\mathrm{d}x\,P_i(x)\bar{Q}_j(x) = \delta_{i,j} - \int_I P_i(x)\bar{Q}_j(x)\,\mathrm{d}x. \tag{2.17}$$

To get the probability $E(r,I)$ that an interval I contains exactly r of the eigenvalues x_j, one can, as usual, introduce an extra variable z in the determinant above and differentiate partially with respect to this variable. Thus

$$E(r,I) = \frac{1}{r!}\left(-\frac{\partial}{\partial z}\right)^r \det\left[\delta_{i,j} - z\int_I P_i(x)\bar{Q}_j(x)\,\mathrm{d}x\right]\bigg|_{z=1}. \tag{2.18}$$

One can then take the limit $n\to\infty$, if one so desires, obtaining a Fredholm determinant whose kernel (see, for example, [1], ch 5) will be $K(x,y) = \lim_{n\to\infty} K_n(x,y)$ over the interval I.

3. Mixed correlation functions

When both p and q are positive, then one cannot replace $\det[\mathrm{e}^{-2cx_iy_j}]$ in equation (1.6) by a single term as we did in section 2. One way out of this difficulty is the following. If we can calculate

$$G(u;v)=\int\prod_{j=1}^{n}(u(\xi_j)v(\eta_j))F(\xi;\eta)\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_n\,\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n \tag{3.1}$$

in a closed form for arbitrary functions $u(\xi)$ and $v(\xi)$, by differentiating functionally several times with respect to $u(x)$ and $v(y)$, we then get ($F(\xi;\eta)$ being symmetric in the variables $\xi_1,\ldots,\xi_n$ and $\eta_1,\ldots,\eta_n$):

$$R_{p,q}(x_1,\ldots,x_p;y_1,\ldots,y_q)=\binom{n}{p}\binom{n}{q}\prod_{j=1}^{p}\frac{\delta}{\delta u(x_j)}\prod_{k=1}^{q}\frac{\delta}{\delta v(y_k)}G(u;v)\bigg|_{u(\xi_j)=v(\eta_j)=1}. \tag{3.2}$$

With the integrations in equation (3.1) being performed over all the variables η_j, we can then replace $\det[\mathrm{e}^{-2c\xi_i\eta_j}]$ by $\exp(-2c\sum_{j=1}^{n}\xi_j\eta_j)$ as in section 2. The difficulty now is that when $u(\xi)$ and $v(\xi)$ are arbitrary functions, which is essential here, one cannot introduce orthogonal polynomials. However, as Dyson remarked [7], we actually do not need the integral (3.1) in any greater detail than its power series expansion when $u(\xi)$ and $v(\xi)$ are near unity. So let us put $u(\xi)=1+a(\xi)$ and $v(\xi)=1+b(\xi)$ and treat $a(\xi)$ and $b(\xi)$ as small quantities.

Therefore,

$$\begin{aligned}G(1+a;1+b)&=f_2n!\int\prod_{j=1}^{n}\{(1+a(\xi_j))(1+b(\eta_j))w(\xi_j,\eta_j)\,\mathrm{d}\xi_j\,\mathrm{d}\eta_j\}\Delta(\xi)\Delta(\eta)\\&\propto\int\prod_{j=1}^{n}\{(1+a(\xi_j))(1+b(\eta_j))w(\xi_j,\eta_j)\,\mathrm{d}\xi_j\,\mathrm{d}\eta_j\}\det[P_{j-1}(\xi_i)]\det[Q_{j-1}(\eta_i)]\end{aligned} \tag{3.3}$$

or, in view of equation (2.6) (and taking proper care of the constants),

$$\begin{aligned}G(1+a;1+b)&=\sum_{(j)}\varepsilon_{(j)}(\delta_{0,j_0}+c_{0,j_0})\ldots(\delta_{n-1,j_{n-1}}+c_{n-1,j_{n-1}})\\&=\det[\delta_{j,k}+c_{j,k}]\end{aligned} \tag{3.4}$$

where $(j)=(j_0,j_1,\ldots,j_{n-1})$ is a permutation of $(0,1,\ldots,n-1)$, $\varepsilon_{(j)}$ is its sign and

$$c_{j,k}=\int w(\xi,\eta)[a(\xi)+b(\eta)+a(\xi)b(\eta)]P_j(\xi)Q_k(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.5}$$

Expanding equation (3.4) in powers of $c_{j,k}$ and collecting terms, one gets

$$\begin{aligned}G(1+a;1+b)&=1+\sum_{j=0}^{n-1}c_{j,j}+\frac{1}{2!}\sum_{j,k=0}^{n-1}\det\begin{bmatrix}c_{j,j}&c_{j,k}\\c_{k,j}&c_{k,k}\end{bmatrix}+\cdots\\&\quad+\frac{1}{k!}\sum_{j_1,j_2,\ldots,j_k=0}^{n-1}\det\begin{bmatrix}c_{j_1,j_1}&\cdots&c_{j_1,j_k}\\\cdots&\cdots&\cdots\\c_{j_k,j_1}&\cdots&c_{j_k,j_k}\end{bmatrix}+\cdots.\end{aligned} \tag{3.6}$$

To obtain the correlation function $R_{p,q}$, one has to differentiate equation (3.6) functionally with respect to $a(x_1), \ldots, a(x_p)$, $b(y_1), \ldots, b(y_q)$ and then put $a(x_j) = b(y_j) = 0$. Terms outside the range $(p+q)/2 \leqslant k \leqslant p+q$ on the right-hand side of equation (3.6) will not contribute to $R_{p,q}$.

Thus,

$$R_{1,1}(x;y) \propto \frac{\delta}{\delta a(x)}\frac{\delta}{\delta b(y)}\left\{\sum_j c_{j,j} + \frac{1}{2!}\sum_{j,k}(c_{j,j}c_{k,k} - c_{j,k}c_{k,j})\right\}\Bigg|_{a(\xi_j)=b(\eta_j)=0}$$
$$= w(x,y)\sum_j P_j(x)Q_j(y) + \frac{1}{2!}\sum_{j,k}\det\begin{bmatrix} P_j(x) & \bar{P}_j(y) \\ P_k(x) & \bar{P}_k(y)\end{bmatrix}\det\begin{bmatrix}\bar{Q}_j(x) & Q_j(y) \\ \bar{Q}_k(x) & Q_k(y)\end{bmatrix} \tag{3.7}$$

where $\bar{Q}_k(x)$ is given by equation (2.7) and $\bar{P}_k(y)$ is given by a similar equation

$$\bar{P}_k(y) = \int P_k(x)w(x,y)\,\mathrm{d}x. \tag{3.8}$$

Note that $P_j(x)$ and $Q_j(x)$ are polynomials, while $\bar{P}_j(x)$ and $\bar{Q}_j(x)$ are, in general, not.

Similarly,

$$R_{2,1}(x_1,x_2;y) \propto \frac{\delta}{\delta a(x_1)}\frac{\delta}{\delta a(x_2)}\frac{\delta}{\delta b(y)}\left\{\frac{1}{2!}\sum_{j,k=0}^{n-1}\det\begin{bmatrix} c_{j,j} & c_{j,k} \\ c_{k,j} & c_{k,k}\end{bmatrix}\right.$$
$$\left. + \frac{1}{3!}\sum_{j,k,\ell=0}^{n-1}\det\begin{bmatrix} c_{j,j} & c_{j,k} & c_{j,\ell} \\ c_{k,j} & c_{k,k} & c_{k,\ell} \\ c_{\ell,j} & c_{\ell,k} & c_{\ell,\ell}\end{bmatrix}\right\}\Bigg|_{a(\xi_j)=b(\eta_j)=0}. \tag{3.9}$$

On rearranging, the first term gives

$$\frac{1}{2!}\sum_{\alpha=1}^{2} w(x_\alpha, y)\sum_{j,k=0}^{n-1}\det\begin{bmatrix} P_j(x_\alpha) & P_j(x_{3-\alpha}) \\ P_k(x_\alpha) & P_k(x_{3-\alpha})\end{bmatrix}\det\begin{bmatrix} Q_j(y) & \bar{Q}_j(x_{3-\alpha}) \\ Q_k(y) & \bar{Q}_k(x_{3-\alpha})\end{bmatrix} \tag{3.10}$$

while the second term gives

$$\frac{1}{3!}\sum_{j,k,\ell=0}^{n-1}\det\begin{bmatrix} P_j(x_1) & P_j(x_2) & \bar{P}_j(y) \\ P_k(x_1) & P_k(x_2) & \bar{P}_k(y) \\ P_\ell(x_1) & P_\ell(x_2) & \bar{P}_\ell(y)\end{bmatrix}\det\begin{bmatrix}\bar{Q}_j(x_1) & \bar{Q}_j(x_2) & Q_j(y) \\ \bar{Q}_k(x_1) & \bar{Q}_k(x_2) & Q_k(y) \\ \bar{Q}_\ell(x_1) & \bar{Q}_\ell(x_2) & Q_\ell(y)\end{bmatrix} \tag{3.11}$$

so that $R_{2,1}(x_1,x_2;y)$ is proportional to the sum of expressions (3.10) and (3.11).

The next correlation function $R_2(x_1,x_2;y_1,y_2)$ is

$$R_{2,2}(x_1,x_2;y_1,y_2) \propto \frac{\delta}{\delta a(x_1)}\frac{\delta}{\delta a(x_2)}\frac{\delta}{\delta b(y_1)}\frac{\delta}{\delta b(y_2)}$$
$$\times\left\{\frac{1}{2!}\det\begin{bmatrix} c_{j,j} & c_{j,k} \\ c_{k,j} & c_{k,k}\end{bmatrix} + \frac{1}{3!}\det\begin{bmatrix} c_{j,j} & c_{j,k} & c_{j,\ell} \\ c_{k,j} & c_{k,k} & c_{k,\ell} \\ c_{\ell,j} & c_{\ell,k} & c_{\ell,\ell}\end{bmatrix}\right.$$
$$\left. + \frac{1}{4!}\det\begin{bmatrix} c_{j,j} & c_{j,k} & c_{j,\ell} & c_{j,m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m,j} & c_{m,k} & c_{m,\ell} & c_{m,m}\end{bmatrix}\right\}\Bigg|_{a(\xi_j)=b(\eta_j)=0} \tag{3.12}$$

The first term gives the contribution

$$\frac{1}{2!}\det\begin{bmatrix} w(x_1,y_1) & w(x_1,y_2) \\ w(x_2,y_1) & w(x_2,y_2)\end{bmatrix}\sum_{j,k}\det\begin{bmatrix} P_j(x_1) & P_j(x_2) \\ P_k(x_1) & P_k(x_2)\end{bmatrix}\det\begin{bmatrix} Q_j(y_1) & Q_j(y_2) \\ Q_k(y_1) & Q_k(y_2)\end{bmatrix}. \tag{3.13}$$

The second term gives

$$\frac{1}{3!}\sum_{\alpha,\beta=1}^{2} w(x_\alpha,y_\beta)\sum_{j,k,\ell=0}^{n-1}\det\begin{bmatrix} P_j(x_{3-\alpha}) & \bar{P}_j(y_{3-\beta}) & P_j(x_\alpha) \\ P_k(x_{3-\alpha}) & \bar{P}_k(y_{3-\beta}) & P_k(x_\alpha) \\ P_\ell(x_{3-\alpha}) & \bar{P}_\ell(y_{3-\beta}) & P_\ell(x_\alpha)\end{bmatrix}$$
$$\times\det\begin{bmatrix} \bar{Q}_j(x_{3-\alpha}) & Q_j(y_{3-\beta}) & Q_j(y_\alpha) \\ \bar{Q}_k(x_{3-\alpha}) & Q_k(y_{3-\beta}) & Q_k(y_\alpha) \\ \bar{Q}_\ell(x_{3-\alpha}) & Q_\ell(y_{3-\beta}) & Q_\ell(y_\alpha)\end{bmatrix}. \tag{3.14}$$

The third term gives

$$\frac{1}{4!}\sum_{j,k,\ell,m=0}^{n-1}\det\begin{bmatrix} P_j(x_1) & P_j(x_2) & \bar{P}_j(y_1) & \bar{P}_j(y_2) \\ P_k(x_1) & P_k(x_2) & \bar{P}_k(y_1) & \bar{P}_k(y_2) \\ P_\ell(x_1) & P_\ell(x_2) & \bar{P}_\ell(y_1) & \bar{P}_\ell(y_2) \\ P_m(x_1) & P_m(x_2) & \bar{P}_m(y_1) & \bar{P}_m(y_2)\end{bmatrix}$$
$$\times\det\begin{bmatrix} \bar{Q}_j(x_1) & \bar{Q}_j(x_2) & Q_j(y_1) & Q_j(y_2) \\ \bar{Q}_k(x_1) & \bar{Q}_k(x_2) & Q_k(y_1) & Q_k(y_2) \\ \bar{Q}_\ell(x_1) & \bar{Q}_\ell(x_2) & Q_\ell(y_1) & Q_\ell(y_2) \\ \bar{Q}_m(x_1) & \bar{Q}_m(x_2) & Q_m(y_1) & Q_m(y_2)\end{bmatrix}. \tag{3.15}$$

$R_{2,2}(x_1,x_2;y_1,y_2)$ is proportional to the sum of the last three terms.

The general procedure, though somewhat lengthy, is now straightforward.

One point to note is that eigenvalues belonging to the same matrix 'repel' each other, while eigenvalues of A do not 'repel' those of B.

One can also obtain the spacing functions of the two sets of eigenvalues. To obtain the probability $E(0,I_1;0,I_2)$ that an interval I_1 does not contain any eigenvalues x_j of matrix A and an interval I_2 does not contain any eigenvalues y_j of matrix B (the intervals I_1 and I_2 may or may not have any overlap), one takes

$$u(x)=\begin{cases}0 & \text{if } x\in I_1\\ 1 & \text{otherwise}\end{cases} \tag{3.16}$$

$$v(y)=\begin{cases}0 & \text{if } y\in I_2\\ 1 & \text{otherwise}\end{cases} \tag{3.17}$$

in equation (3.1). This gives

$$E(0,I_1;0,I_2)\propto\frac{1}{n!}\int\prod_{j=1}^{n}\{u(x_j)v(y_j)w(x_j,y_j)\,\mathrm{d}x_j\,\mathrm{d}y_j\}\det[P_{j-1}(x_i)]\det[Q_{j-1}(y_i)]$$
$$=\det\left[\hat{G}_{ij}(I_1;I_2)\right] \tag{3.18}$$

where

$$\hat{G}_{ij}(I_1;I_2)=\int P_i(x)Q_j(y)w(x,y)u(x)v(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=\delta_{i,j}-\int_{I_1}\mathrm{d}x\,P_i(x)\bar{Q}_j(x)-\int_{I_2}\mathrm{d}y\,\bar{P}_i(y)Q_j(y)$$
$$+\int_{I_1}\mathrm{d}x\int_{I_2}\mathrm{d}y\,P_i(x)Q_j(y)w(x,y). \tag{3.19}$$

If one wants to obtain the probability $E(r, I_1; s, I_2)$ that an interval I_1 contains exacly r eigenvalues x_j of matrix A and an interval I_2 contains exacly s eigenvalues y_j of matrix B, then one introduces two extra variables z_1 and z_2 in equation (3.19)

$$\hat{G}_{ij}(z_1, I_1; z_2, I_2) = \delta_{i,j} - z_1 \int_{I_1} \mathrm{d}x\, P_i(x)\bar{Q}_j(x) - z_2 \int_{I_2} \mathrm{d}y\, \bar{P}_i(y)Q_j(y) + z_1 z_2 \int_{I_1} \mathrm{d}x \int_{I_2} \mathrm{d}y\, P_i(x)Q_j(y)w(x, y) \tag{3.20}$$

so that

$$E(r, I_1; s, I_2) = \frac{1}{r!}\left(-\frac{\partial}{\partial z_1}\right)^r \frac{1}{s!}\left(-\frac{\partial}{\partial z_2}\right)^s \det\left[\hat{G}_{ij}(z_1, I_1; z_2, I_2)\right]\Big|_{z_1=z_2=1}. \tag{3.21}$$

To take the limit $n \to \infty$, it will be convenient to write $\hat{G}_{ij}$ as

$$\hat{G}_{ij} = \delta_{i,j} - z_1 \int_{I_1} \mathrm{d}x\, P_i(x)\bar{Q}_j(x) - z_2 \int_{-\infty}^{\infty} \mathrm{d}x \int_{I_2} \mathrm{d}y\, P_i(x)w(x, y)Q_j(y) + z_1 z_2 \int_{I_1} \mathrm{d}x \int_{I_2} \mathrm{d}y\, P_i(x)w(x, y)Q_j(y). \tag{3.22}$$

Now consider the integral equation

$$\lambda f(x) = z_1 \int_{I_1} [K_n(x, y) - z_2\bar{K}_n(x, y)] f(y)\,\mathrm{d}y + z_2 \int_{-\infty}^{\infty} \bar{K}_n(x, y) f(y)\,\mathrm{d}y \tag{3.23}$$

where $K_n(x, y)$ is given by equation (2.9), and

$$\bar{K}_n(x, y) = \sum_{i=0}^{n-1} P_i(x) \int_{I_2} Q_i(\xi)w(y, \xi)\,\mathrm{d}\xi. \tag{3.24}$$

The eigenfunction $f(x)$ is necessarily of the form

$$f(x) = \sum_{i=0}^{n-1} k_i P_i(x) \tag{3.25}$$

with certain constants k_i. Substituting (3.25) into equation (3.23) and remembering that $P_i(x)$ are linearly independent polynomials, we get

$$\sum_j \left\{\lambda\delta_{i,j} - z_1 \int \mathrm{d}y\, \bar{Q}_i(y)P_j(y) - z_2 \int_{-\infty}^{\infty} \mathrm{d}y\, P_j(y) \int_{I_2} \mathrm{d}\xi\, Q_i(\xi)w(y, \xi) + z_1 z_2 \int_{I_1} \mathrm{d}y\, P_j(y) \int_{I_2} \mathrm{d}\xi\, Q_i(\xi)w(y, \xi)\right\} k_j = 0. \tag{3.26}$$

Equation (3.26) has a non-zero solution if the coefficient matrix on the left-hand side is singular. This gives an algebraic equation of order n in λ having n roots $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$; so that one can write

$$\det[(\lambda - 1)\delta_{i,j} + \hat{G}_{ij}(z_1, I_1; z_2, I_2)] = \prod_{i=0}^{n-1}(\lambda - \lambda_i). \tag{3.27}$$

Setting $\lambda = 1$ in (3.27), we finally get

$$\det\left[\hat{G}_{ij}\right] = \prod_{i=0}^{n-1}(1 - \lambda_i). \tag{3.28}$$

Thus, to obtain the limit of $\det[\hat{G}_{ij}]$ as $n \to \infty$, one has to take the limits of $K_n(x, y)$ and $\bar{K}_n(x, y)$ and then study limiting integral equation (3.23).

The same method can be used to find correlation functions of eigenvalues of three or more matrices coupled in a chain. For example, if equation (1.4) is replaced by

$$\exp[-\operatorname{tr}\{U(A) + V(B) + W(C) + 2c_1AB + 2c_2BC\}] \tag{1.4$'$}$$

then one has only to replace equations (2.3) and (2.6) by

$$w(x, y, z) = \exp[-U(x) - V(y) - W(z) - 2c_1xy - 2c_2yz] \tag{2.3$'$}$$

and

$$\int w(x, y, z)P_j(x)Q_k(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \delta_{j,k}. \tag{2.6$'$}$$

To obtain the correlation functions, replace $c_{j,k}$ in equation (3.5) by

$$c_{j,k} = \int w(x, y, z)[(1 + a_1(x))(1 + a_2(y))(1 + a_3(z)) - 1]P_j(x)Q_k(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{3.5$'$}$$

differentiate functionally with respect to a_1, a_2, a_3 and, finally, set $a_1(x_j) = a_2(y_j) = a_3(z_j) = 0$. Similarly, to get the spacing functions, one replaces equation (3.21) by

$$E(r_1, I_1; r_2, I_2; r_3, I_3) = \frac{1}{r_1!}\left(\frac{-\partial}{\partial z_1}\right)^{r_1}\frac{1}{r_2!}\left(\frac{-\partial}{\partial z_2}\right)^{r_2}\frac{1}{r_3!}\left(\frac{-\partial}{\partial z_3}\right)^{r_3}\det[G_{ij}(z_1, I_1; z_2, I_2; z_3, I_3)]|_{z_1=z_2=z_3=1} \tag{3.21$'$}$$

with an appropriate expression for $G_{ij}(z_1, I_1; z_2, I_2; z_3, I_3)$.

4. A chain of coupled matrices

Consider a set of matrices coupled in a chain, i.e. let the joint probability density of the elements of p $n \times n$ Hermitian matrices $A_1, A_2, \ldots, A_p$ be proportional to

$$\exp\left\{-\sum_{j=1}^{p}\operatorname{tr}[V_j(A_j) + \alpha_jA_jA_{j+1}]\right\} \qquad \alpha_p \equiv 0. \tag{4.1}$$

Then the joint probability density of the set of their eigenvalues can be deduced to be (see, for example, [5], ch 13)

$$\exp\left\{-\sum_{j=1}^{p}\sum_{k=1}^{n}V_j(x_{j,k})\right\}\prod_{j=1}^{p-1}\det[\exp(-\alpha_jx_{j,k}x_{j+1,\ell})]_{k,\ell=1,\ldots,n}\Delta(x_1)\Delta(x_p) \tag{4.2}$$

where $x_{j,1}, \ldots, x_{j,n}$ are the eigenvalues of A_j.

If one is interested in the correlations of the eigenvalues of one particular matrix, say A_2, disregarding the eigenvalues of all the other matrices, then a little reflection over the anti-symmetry of various factors will show that one can replace each of the determinants $\det[\exp(-\alpha_j x_{j,k} x_{j+1,\ell})]$ by its diagonal term. In other words, for the correlations among the eigenvalues of any one particular matrix, disregarding those of others, it is permissible to replace expression (4.2) by

$$(n!)^{p-1} \exp\left\{-\sum_{k=1}^{n}\sum_{j=1}^{p}[V_j(x_{j,k}) + \alpha_j x_{j,k} x_{j+1,k}]\right\}\Delta(x_1)\Delta(x_p). \tag{4.3}$$

One will then choose polynomials $P_j(x)$ and $Q_j(x)$ such that

$$\int \exp\left\{-\sum_{q=1}^{p}[V_q(y_q) + \alpha_q y_q y_{q+1}]\right\} P_j(y_1) Q_k(y_p)\,\mathrm{d}y_1 \ldots \mathrm{d}y_p = \delta_{j,k}. \tag{4.4}$$

To study the correlations of the eigenvalues of matrix A_2, say, one then defines

$$\hat{P}_k(x) = \int \exp\{-V_1(y) - V_2(x) - \alpha_1 y x\} P_k(y)\,\mathrm{d}y \tag{4.5}$$

and

$$\hat{Q}_k(x) = \int \exp\{-V_3(y_3) - \cdots - V_p(y_p) - \alpha_2 x y_3 - \alpha_3 y_3 y_4 - \cdots - \alpha_{p-1} y_{p-1} y_p\} \times Q_k(y_p)\,\mathrm{d}y_3 \ldots \mathrm{d}y_p. \tag{4.6}$$

These $\hat{P}_k(x)$ and $\hat{Q}_k(x)$ are no longer polynomials in general, but they are orthogonal, and can be used with as much convenience. Thus, for example, the m-point correlation function for the eigenvalues of matrix A_2 is

$$R_m(x_{2,1}, \ldots, x_{2,m}) = \det\left[\hat{K}_n(x_{2,i}, x_{2,j})\right]_{i,j=1,\ldots,m} \tag{4.7}$$

with

$$\hat{K}_n(x, y) = \sum_{j=0}^{n-1} \hat{P}_j(x)\hat{Q}_j(y). \tag{4.8}$$

The spacing functions are similarly given by the partial derivatives of the determinant

$$\det\left[\delta_{i,j} - z\int_J \hat{P}_i(x)\hat{Q}_j(x)\,\mathrm{d}x\right] \tag{4.9}$$

see equation (2.18).

If one is interested in the mixed correlations or in the spacing functions of eigenvalues of (any) two matrices in the coupled chain of p matrices, a similar procedure reduces it to the study of correlations of two coupled matrices from section 3.

5. An example

Consider the case $V(x) = x^2$. Equation (1.5) now takes the form

$$F(x; y) = f_2 \exp\left\{ -\sum_{j=1}^{n} \left[U(x_j) + y_j^2\right] \right\} \det[-2cx_i y_j]\Delta(x)\Delta(y). \tag{5.1}$$

Since for any polynomial $Q_k(y)$ of degree k, $\int \exp[-(y+cx)^2]Q_k(y)\,dy$ is also a polynomial of the same degree k, equation (2.6) can be replaced by

$$\int \exp\left(-U(x) + c^2x^2\right) P_j(x)P_k(x)\,dx = \delta_{j,k}. \tag{5.2}$$

Thus, $P_j(x)$ are orthogonal polynomials with weight $\exp[-U(x) + c^2x^2]$

$$\bar{Q}_j(x) = \exp\left[-U(x) + c^2x^2\right] P_j(x) \tag{5.3}$$

and

$$K_n(x, y) = \exp\left[-U(x) + c^2x^2\right] \sum_{j=0}^{n-1} P_j(x)P_j(y). \tag{5.4}$$

However, the expressions for $\bar{P}_j(x)$, $\bar{K}_n(x, y)$ and correlation functions $R_{p,q}$ for $p, q > 0$ are not much simpler.

Moreover, if $U(x) = x^2$, then $P_j(x) \propto H_j(x\sqrt{1-c^2})$, where H_j are Hermite polynomials, and the calculation can be completed.

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